Stochastics and Quality Control

Quality Control Using Convolutional Neural Networks Applied to Samples of Very Small Size

Supplement 2

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1. Formalisms and Notation

1.1. Abbreviations

QC: quality control
NN: artificial neural network
DNN: deep neural network
CNN: convolutional neural network
1-D: one dimensional
GA: genetic algorithms
QCMT: quality control measurements tuple

1.2. Operators

 # : cardinality
~ : distributed as

1.3. Tuples

$\boldsymbol{x}$: *n*-tuple $(x_1, x_2, \dots, x_n)$ of QC measurements

1.4. Datasets

$\mathbf{Z_\theta} = \{\boldsymbol{z}_i\}_{i=1}^n = \{\boldsymbol{x}_i, \mathrm{y}_i\}_{i=1}^n = \{\boldsymbol{x}_i, f(\boldsymbol{x}_i; \boldsymbol{\theta})\}_{i=1}^n$: dataset of *n* -tuples $\boldsymbol{x_i}$ of QC measurements with $\mathrm{y_i} \in \{True, False\}$

$\mathbf{T}(n; \alpha)$: training dataset of *n*-tuples $\boldsymbol{x}_i$ of QC measurements consistıng of:

(1) $a\ 2^{n-1}10^5$ *n*-tuples of random simulated QC measurements distributed as $\mathcal{N}(0,1)$,

(2) $2^{n-1}10^5$ *n*-tuples of random simulated QC measurements, distributed as $\mathcal{N}(0, s^2)$ where $s_0 < s \leq 11s_0$, and

(3) $2^{n-1}\ 10^5$ *n*-tuples of random simulated QC measurements, distributed as $\mathcal{N}(m, 1)$, for $0 < |m| \leq 10s_0$.

$\mathbf{D}(n; m, s)$*:* testing dataset of *n* -tuples $\boldsymbol{x}_i$ of QC measurements, distributed as $\mathcal{N}(m, s^2)$.

1.5. Functions

$\boldsymbol{x}$: *n*-tuple $(x_1, x_2, \dots, x_n)$ of QC measurements

$\mathcal{N}(m, s^2)$: normal distribution with mean *m* and standard deviation *s*.

$N(\boldsymbol{x}; \boldsymbol{\theta}) = R_N(\boldsymbol{x}; m_0, s_0, n, \alpha)$: NN QC function applied to a tuple $\boldsymbol{x}$ of QC measurements, trained using the dataset $\mathbf{T}(n; m_0, s_0, \alpha)$; $N(\boldsymbol{x}; \boldsymbol{\theta}) \in \{True, False\}$

The index α is related to the ratio of the simulated QC measurements of the training set, distributed as $\mathcal{N}(m_0, {s_0}^2)$ and $\mathcal{N}(m, s^2)$, for $0 < |m - m_0| < 8s_0$ or $s_0 < s < 9s_0$.

$R_\mathrm{V}(\boldsymbol{x};\ \boldsymbol{\theta}) = R_V(\boldsymbol{x}; m_0, s_0, n, k, l_k)$: statistical QC function with decision limits $m_0 \pm l_k\ s_0$, applied on a tuple $\boldsymbol{x}$, of *n* control measurements; $R_V(\boldsymbol{x}; m_0, s_0, n, k, l_k) :=$

$$\begin{cases} True \text{ if for } k\ x_i \in \boldsymbol{x}, x_i < m_0 - l_k\ s_0 > \vee\ x_i > m_0 + l_k\ s_0 \\ False \text{ otherwise} \end{cases}$$

$R_M(\boldsymbol{x};\ \boldsymbol{\theta}) = R_M(\boldsymbol{x}; m_0, s_0, n, l_M)$: statistical QC function with decision limits $m_0 \pm l_M s_0$, applied on a tuple $\boldsymbol{x}$, of *n* control measurements; $R_M(\boldsymbol{x}; m_0, s_0, n, l_M) :=$

$\begin{cases} True \text{ if } \bar{x} < m_0 - l_M\, s_0 \vee \bar{x} > m_0 + l_M\, s_0 \\ False \text{ otherwise} \end{cases}$

$R_S(\boldsymbol{x};\, \boldsymbol{\theta}) = R_S(\boldsymbol{x}; m_0, s_0, n, l_S)$: statistical QC function with decision limit $l_S s_0$, applied on a tuple $\boldsymbol{x}$, of *n* control measurements; $R_S(\boldsymbol{x}; m_0, s_0, n, l_S) :=$

$\begin{cases} True \text{ if for } x_i \in \boldsymbol{x}, \sqrt{\frac{\sum_{i=1}^{n} {x_i}^2}{n}} > l_S\, s_0 \\ False \text{ otherwise} \end{cases}$

$R_{\mathrm{R}}(\boldsymbol{x};\, \boldsymbol{\theta}) = R_R(\boldsymbol{x}; m_0, s_0, n, l_R)$: statistical QC function with decision limit $l_R s_0$, applied on a tuple $\boldsymbol{x}$, of *n* control measurements; $R_M(\boldsymbol{x}; m_0, s_0, n, l_R) :=$

$\begin{cases} True \text{ if } max(\boldsymbol{x}) - min(\boldsymbol{x}) > l_R s_0 \\ False \text{ otherwise} \end{cases}$

$L(f_q; \boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{I}\left(f_q(\boldsymbol{x}_i; \boldsymbol{\theta}) \neq f(\boldsymbol{x}_i; \boldsymbol{\theta})\right)$: misclassification rate of QC function f_q, where $I(e) = \begin{cases} 1 \text{ if } e\ True, \\ 0 \text{ if } e\ False \end{cases}$,

$f_q(\boldsymbol{x}; \boldsymbol{\theta}) = \begin{cases} True \text{ if } \exists\, \mathrm{x_i} \in \boldsymbol{x}, X_i \sim \mathcal{N}(m, s^2) \wedge (\mathrm{m} > 0 \vee \mathrm{s} > 1), \\ False \text{ if } \forall\, x_i \in \boldsymbol{x}, X_i \sim \mathcal{N}(0,1) \end{cases}$ and $f_q(\boldsymbol{x};\, \boldsymbol{\theta}) \in$

$\{N(\boldsymbol{x}; \boldsymbol{\theta}), R_V(\boldsymbol{x};\, \boldsymbol{\theta}), R_{\mathrm{M}}(\boldsymbol{x};\, \boldsymbol{\theta}), R_{\mathrm{S}}(\boldsymbol{x};\, \boldsymbol{\theta}), R_{\mathrm{R}}(\boldsymbol{x};\, \boldsymbol{\theta})\}$

$P_N(.) = P_N(\mathrm{m}, \mathrm{s}; m_0, s_0, n, a)$: probability for rejection of the *QC* function $R_N(\boldsymbol{x}; m_0, s_0, n, a)$, applied to a *n* -tuples of control measurements, distributed as $\mathcal{N}(m, s^2)$.

$P_V(.) = P_V(m, s; m_0, s_0, n, k, l_k)$: probability for rejection of the QC function $R_V(\boldsymbol{x}; m_0, s_0, n, k, l_k)$ applied on a *n* -tuples of control measurements, distributed as $\mathcal{N}(m, s^2)$

$P_M(.) = P_M(\mathrm{m}, \mathrm{s}; m_0, s_0, n, l_M)$: probability for rejection of the QC function $R_M(\boldsymbol{x}; m_0, s_0, n, l_M)$ applied on *n* -tuples of control measurements, distributed as $\mathcal{N}(m, s^2)$

$P_S(.) = P_S(\mathrm{m}, \mathrm{s}; m_0, s_0, n, l_S)$: probability for rejection of the QC function $R_S(\boldsymbol{x}; m_0, s_0, n, l_S)$ applied on *n* -tuples of control measurements, distributed as $\mathcal{N}(m, s^2)$

$P_R(.) = P_R(\mathrm{m}, \mathrm{s}; m_0, s_0, n, l_R)$: probability for rejection of the QC function $R_R(\boldsymbol{x}; m_0, s_0, n, k, l_R)$ applied on *n* -tuples of control measurements, distributed as $\mathcal{N}(m, s^2)$

$$\Delta P_{N,V}(.) = \Delta P_{N,V}(m, s; m_0, s_0, n, k, \alpha)$$
$$= P_N(m, s; m_0, s_0, n, \alpha) - P_V(m, s; m_0, s_0, n, k, l_k)$$
$$\Delta P_{N,M}(.) = P_N(m, s; m_0, s_0, n, \alpha) - P_M(m, s; m_0, s_0, n, l_M)$$
$$\Delta P_{N,S}(.) = P_N(m, s; m_0, s_0, n, \alpha) - P_S(m, s; m_0, s_0, n, l_S)$$
$$\Delta P_{N,R}(.) = P_N(m, s; m_0, s_0, n, \alpha) - P_R(m, s; m_0, s_0, n, l_R)$$

$\binom{n}{k}$: binomial coefficient

erf(z): error function

erfc(z): complementary error function

$\Gamma(z)$: gamma function

$\Gamma(a, z_0, z_1)$: generalized incomplete gamma function

${}_2F_1(a, b; c; z)$: hypergeometric function

1.6.Computations

a) Using k^{th} order statistics and the folded normal distribution (David and Ngaraja 2014; Read 2014), the probability for rejection of the QC function $R_V(\boldsymbol{x}; 0,1, n, k, l)$ is calculated as follows:

$$P_V(m, s; 0,1, n, k, l_k)$$
$$= \frac{1}{k!\,(n-k)!} 2^{-n} \left(erf\left(\frac{m + l_k}{\sqrt{2}s}\right) - erf\left(\frac{m - l_k}{\sqrt{2}s}\right)\right)^{n-k} \left(erf\left(\frac{m - l_k}{\sqrt{2}}\right) - erf\left(\frac{m + l_k}{\sqrt{2}}\right)\right)^{k} \Gamma(1+n)\ {}_2F_1\left(1, k-n; 1+k; 1 + \frac{2}{\operatorname{erf}\left(\frac{m - l_k}{\sqrt{2}s}\right) - erf\left(\frac{m + l_k}{\sqrt{2}s}\right)}\right)$$

To define l_k, we solve the following equation for $a \in \{1,2,3,4,6,8,12,16\}$, $2 \leq n \leq 4$ and $1 \leq k \leq n$:

$$P_V(0,1; 0,1, n, k, l_k) = P_N(0,1; 0,1, n, \alpha)$$

b) The probability for rejection for the standardized random error *s* and the systematic error *m* of the statistical QC function $R_M(\boldsymbol{x}; 0,1, n, k, l_S)$ is calculated as follows, using the distribution of the sample mean (Solomon and Stephens 2014):

$$P_M(m,s;0,1,n,l_M) = \frac{1}{2}\left(2 - erf\left(\frac{(m+l_M)\sqrt{n}}{\sqrt{2}s}\right) + erf\left(\frac{(m-l_M)\sqrt{n}}{\sqrt{2}s}\right)\right)$$

To define l_M, we solve the following equation for $a \in \{1,2,3,4,6,8,12,16\}$, and $2 \leq n \leq 4$:

$$P_M(0,1;0,1,n,l_M) = P_N(0,1;0,1,n,\alpha)$$

c) If s^2 the variance of a sample of size n and σ^2 the variance of the population, the ratio $\frac{(n-1)s^2}{\sigma^2}$ is χ^2 distributed, with $(n-1)$ degrees of freedom (Solomon and Stephens 2014). Accordingly, it can be shown that the probability for rejection of the QC function $R_S(\boldsymbol{x};0,1,n,k,l_S)$ is calculated as follows:

$$P_S(m,s;0,1,n,l_S) = 1 - \frac{\Gamma\left(\frac{1}{2}(n-1),0,\frac{{l_S}^2(n-1)}{2s^2}\right)}{\Gamma\left(\frac{1}{2}(n-1)\right)}$$

To define l_S, we solve the following equation for $a \in \{1,2,3,4,6,8,12,16\}$, and $2 \leq n \leq 4$:

$$P_S(0,1;0,1,n,l_S) = P_N(0,1;0,1,n,\alpha)$$

d) Using the distribution of the range of the values of samples (David and Ngaraja 2014), the probabilities for rejection of the QC function $R_R(\boldsymbol{x};0,1,n,k,l_R)$ are calculated as follows:

$$P_R(m,s;0,1,n,l_R) = 1 - n\int_{-\infty}^{\infty} \frac{2^{n-\frac{1}{2}}\, e^{-\frac{x^2}{\sqrt{2}}}\left(erf\left(\frac{\frac{l_R}{s}+x}{\sqrt{2}}\right) - erf\left(\frac{x}{\sqrt{2}}\right)\right)^{n-1}}{\sqrt{\pi}}\, dx$$

To define l_R, we solve the following equation for $a \in \{1,2,3,4,6,8,12,16\}$, and $2 \leq n \leq 4$:

$$P_R(0,1;0,1,n,l_R) = P_N(0,1;0,1,n,\alpha)$$

Also, for $a \in \{1,2,3,4,6,8,12,16\}$, $2 \leq n \leq 4$ and $1 \leq k \leq n$:

$$\Delta P_{N,V}(m,s;0,1,n,k,\alpha) = P_N(m,s;0,1,n,\alpha) - P_V(m,s;0,1,n,k,l_k)$$

$$\Delta P_{N,M}(m,s;0,1,n,\alpha) = P_N(m,s;0,1,n,\alpha) - P_M(m,s;0,1,n,l_M)$$

$$\Delta P_{N,S}(m,s;0,1,n,\alpha) = P_N(m,s;0,1,n,\alpha) - P_S(m,s;0,1,n,l_S)$$

$$\Delta P_{N,R}(m,s;0,1,n,\alpha) = P_N(m,s;0,1,n,\alpha) - P_R(m,s;0,1,n,l_R)$$

2. Tables

Table 1. The calculated values of l_k and the respective probabilities for false rejection $P_V(0,1;0,1,n,k,l_k) = P_N(0,1;0,1,n,a)$ for $a \in \{1,2,3,4\}, 2 \leq n \leq 4$ and $1 \leq k \leq n$.

a	n	k	l_k	$P_V(0,1;0,1,n,k,l_k)$
1	2	1	1.88809	0.114545
		2	0.957242	
	3	1	2.033406	0.120814
		2	1.234588	
		3	0.683399	
	4	1	2.220312	0.101483
		2	1.462023	
		3	0.989899	
		4	0.576298	
2	2	1	2.268308	0.046078
		2	1.240862	
	3	1	2.456273	0.041528
		2	1.543181	
		3	0.941802	
	4	1	2.56916	0.040159
		2	1.712432	
		3	1.20096	
		4	0.759328	
3	2	1	2.407227	0.031890
		2	1.345147	
	3	1	2.646056	0.024233
		2	1.680857	
		3	1.059486	
	4	1	2.681327	0.029011
		2	1.792298	
		3	1.26849	
		4	0.81914	
4	2	1	2.537022	0.008733
		2	1.442668	
	3	1	2.814931	0.007836
		2	1.802939	
		3	1.164519	
	4	1	2.90121	0.004909
		2	1.948125	
		3	1.400324	
		4	0.937094	

Table 2. The calculated values of l_k and the respective probabilities for false rejection $P_V(0,1;0,1,n,k,l_k) = P_N(0,1;0,1,n,a)$ for $a \in \{6,8,12,16\}$, $2 \leq n \leq 4$ and $1 \leq k \leq n$.

a	n	k	l_k	$P_V(0,1;0,1,n,k,l_k)$
6	2	1	2.646416	0.016204
		2	1.52486	
	3	1	2.958895	0.009234
		2	1.906731	
		3	1.254133	
	4	1	3.010815	0.010381
		2	2.025504	
		3	1.465777	
		4	0.996104	
8	2	1	2.909426	0.009049
		2	1.722262	
	3	1	3.13503	0.005146
		2	2.033421	
		3	1.363742	
	4	1	3.194108	0.005599
		2	2.154577	
		3	1.574881	
		4	1.094944	
12	2	1	3.145682	0.003311
		2	1.89916	
	3	1	3.131339	0.005211
		2	2.030769	
		3	1.361446	
	4	1	3.301042	0.003848
		2	2.229736	
		3	1.638351	
		4	1.152649	
16	2	1	3.218706	0.002574
		2	1.953739	
	3	1	3.298332	0.002915
		2	2.150646	
		3	1.465261	
	4	1	3.284519	0.004080
		2	2.218128	
		3	1.628552	
		4	1.143732	

Table 3. The calculated values of l_M, l_S, and l_R, and the respective probabilities for false rejection $P_M(0{,}1;0{,}1,n,l_M) = P_S(0{,}1;0{,}1,n,l_S) = P_R(0{,}1;0{,}1,n,l_R) = P_N(0{,}1;0{,}1,n,a)$ for $a \in \{1,2,3,4,6,8,12,16\}$, and $2 \leq n \leq 4$.

a	n	l_M	l_S	l_R	$P_N(0{,}1;0{,}1,n,a)$
1	2	1.115878	1.578090	2.231756	0.114545
	3	0.895682	1.453790	2.779737	0.120814
	4	0.818854	1.439650	3.231524	0.101483
2	2	1.410454	1.994683	2.820907	0.046078
	3	1.176769	1.783648	3.416732	0.041528
	4	1.026056	1.663569	3.747944	0.040159
3	2	1.517300	2.145787	3.034601	0.031890
	3	1.301014	1.928746	3.697987	0.024233
	4	1.091667	1.734083	3.911540	0.029011
4	2	1.616679	2.286329	3.233358	0.008733
	3	1.410477	2.056486	3.946152	0.007836
	4	1.218786	1.870495	4.229342	0.004909
6	2	1.700092	2.404294	3.400185	0.114545
	3	1.503002	2.164461	4.156331	0.120814
	4	1.281432	1.937687	4.386514	0.101483
8	2	1.899372	2.686118	3.798745	0.046078
	3	1.615272	2.295536	4.411990	0.041528
	4	1.385200	2.049009	4.647825	0.040159
12	2	2.076957	2.937261	4.153915	0.031890
	3	1.612930	2.292801	4.406648	0.024233
	4	1.445201	2.113418	4.799526	0.029011
16	2	2.131597	3.014533	4.263194	0.008733
	3	1.718519	2.416172	4.647779	0.007836
	4	1.435955	2.103490	4.776119	0.004909

Table 4. The probabilities for false rejection $P_N(0,1;0,1,n,a)$, and for critical random error detection $P_N(0,s_c;0,1,n,a)$ and $P_V(0,s_c;0,1,n,k,l_k)$, for $2 \leq n \leq 4$, $1 \leq k \leq n$, $a \in \{1,2,3,4,6,8,12,16\}$, and $s_c = 3.32$, given that $P_N(0,1;0,1,n,a) = P_V(0,1;0,1,n,k,l_k)$.

a	n	$P_N(0,1;0,1,n,a)$	$P_N(0,3.32;0,1,n,a)$	$P_V(0,3.32;0,1,n,1,l_1)$	$P_V(0,3.32;0,1,n,2,l_2)$	$P_V(0,3.32;0,1,n,3,l_3)$	$P_V(0,3.32;0,1,n,4,l_4)$
1	2	0.114545	0.819895	0.814675	0.597633		
	3	0.120814	0.909142	0.902772	0.796424	0.586149	
	4	0.101483	0.948875	0.939273	0.882532	0.764228	0.552561
2	2	0.046078	0.752273	0.744375	0.502043		
	3	0.041528	0.855796	0.841954	0.707301	0.468427	
	4	0.040159	0.916656	0.900923	0.827587	0.682441	0.450069
3	2	0.031890	0.727635	0.717348	0.469658		
	3	0.024233	0.828505	0.810273	0.66606	0.421202	
	4	0.029011	0.906610	0.886242	0.808196	0.655873	0.420129
4	2	0.022235	0.700472	0.691651	0.440703		
	3	0.014565	0.803876	0.780139	0.629244	0.382236	
	4	0.014786	0.881334	0.854258	0.768174	0.604101	0.365836
6	2	0.016204	0.683072	0.669746	0.417287		
	3	0.009234	0.779195	0.753200	0.597984	0.351265	
	4	0.010381	0.868898	0.836804	0.747369	0.578624	0.340919
8	2	0.007229	0.635205	0.616572	0.364675		
	3	0.005146	0.749863	0.718935	0.560122	0.316105	
	4	0.005599	0.846729	0.805545	0.711545	0.536766	0.302330
12	2	0.003311	0.591526	0.568773	0.321767		
	3	0.005211	0.748990	0.719665	0.560910	0.316811	
	4	0.003848	0.833371	0.786207	0.690159	0.512875	0.281529
16	2	0.002574	0.573138	0.554090	0.309314		
	3	0.002915	0.722918	0.686142	0.525591	0.286088	
	4	0.004080	0.828561	0.789244	0.693483	0.516538	0.284663

Table 5. The probabilities for false rejection $P_N(0,1;0,1,n,a)$, and for critical random error detection $P_N(0,s_c;0,1,n,a)$, $P_M(0,s_c;0,1,n,k,l_M)$, $P_S(0,s_c;0,1,n,k,l_S)$, and $P_R(0,s_c;0,1,n,k,l_R)$, for $2 \leq n \leq 4$, $a \in \{1,2,3,4,6,8,12,16\}$, and $s_c = 3.32$, given that $P_N(0,1;0,1,n,a) = P_M(0,1;0,1,n,l_M) = P_S(0,1;0,1,n,l_S) = P_R(0,1;0,1,n,l_R)$.

a	n	$P_N(0,1;0,1,n,a)$	$P_N(0,3.32;0,1,n,a)$	$P_M(0,3.32;0,1,n,l_M)$	$P_S(0,3.32;0,1,n,l_S)$	$P_R(0,3.32;0,1,n,l_R)$
1	2	0.114545	0.819895	0.634507	0.634507	0.634507
	3	0.120814	0.909142	0.640255	0.825474	0.824353
	4	0.101483	0.948875	0.621764	0.904564	0.901588
2	2	0.046078	0.752273	0.547914	0.547914	0.547914
	3	0.041528	0.855796	0.539212	0.749231	0.746981
	4	0.040159	0.916656	0.536451	0.860570	0.855202
3	2	0.031890	0.727635	0.518015	0.518015	0.518015
	3	0.024233	0.828505	0.497244	0.713487	0.710594
	4	0.029011	0.906610	0.510719	0.845000	0.838689
4	2	0.022235	0.700472	0.490983	0.490983	0.490983
	3	0.014565	0.803876	0.461762	0.681275	0.677749
	4	0.014786	0.881334	0.462762	0.812738	0.804351
6	2	0.016204	0.683072	0.468892	0.468892	0.468892
	3	0.009234	0.779195	0.432908	0.653673	0.649568
	4	0.010381	0.868898	0.440084	0.795884	0.786356
8	2	0.007229	0.635205	0.418411	0.418411	0.418411
	3	0.005146	0.749863	0.399338	0.619898	0.615046
	4	0.005599	0.846729	0.403960	0.766703	0.755130
12	2	0.003311	0.591526	0.376245	0.376245	0.376245
	3	0.005211	0.748990	0.400022	0.620605	0.615769
	4	0.003848	0.833371	0.383907	0.749170	0.736334
16	2	0.002574	0.573138	0.363818	0.363818	0.363818
	3	0.002915	0.722918	0.369892	0.588734	0.583162
	4	0.004080	0.828561	0.386957	0.751901	0.739264

Table 6. The probabilities for false rejection $P_N(0,1;0,1,n,a)$, and for critical systematic error detection $P_N(m_c,1;0,1,n,a)$ and $P_S(m_c,1;0,1,n,k,l_k)$, for $2 \leq n \leq 4$, $1 \leq k \leq n$, $a \in \{1,2,3,4,6,8,12,16\}$, and $m_c = 2.63$, given that $P_N(0,1;0,1,n,a) = P_V(0,1;0,1,n,k,l_k)$.

a	n	$P_N(0,1;0,1,n,a)$	$P_N(2.63,1;0,1,n,a)$	$P_V(2.63,1;0,1,n,1,l_1)$	$P_V(2.63,1;0,1,n,2,l_2)$	$P_V(2.63,1;0,1,n,3,l_3)$	$P_V(2.63,1;0,1,n,4,l_4)$
1	2	0.114545	0.965929	0.946861	0.907276		
	3	0.120814	0.993068	0.978749	0.98088	0.925119	
	4	0.101483	0.998901	0.986195	0.993346	0.985523	0.924058
2	2	0.046078	0.924509	0.869983	0.840762		
	3	0.041528	0.973308	0.918863	0.946975	0.868342	
	4	0.040159	0.993895	0.947949	0.979607	0.967878	0.882843
3	2	0.031890	0.893212	0.828838	0.809597		
	3	0.024233	0.959429	0.868662	0.921011	0.834325	
	4	0.029011	0.993239	0.925541	0.971853	0.959347	0.866581
4	2	0.022235	0.854991	0.783906	0.777096		
	3	0.014565	0.942030	0.809660	0.890711	0.799248	
	4	0.014786	0.985589	0.862698	0.949684	0.937317	0.830015
6	2	0.016204	0.835627	0.741473	0.747237		
	3	0.009234	0.911144	0.749133	0.858957	0.765756	
	4	0.010381	0.975338	0.821382	0.934399	0.923294	0.809382
8	2	0.007229	0.767347	0.625611	0.667052		
	3	0.005146	0.881898	0.664432	0.812332	0.720443	
	4	0.005599	0.967433	0.738234	0.901263	0.894655	0.771242
12	2	0.003311	0.664467	0.511917	0.586912		
	3	0.005211	0.872698	0.666303	0.813397	0.721439	
	4	0.003848	0.956170	0.682879	0.877081	0.874696	0.746913
16	2	0.002574	0.634268	0.476443	0.561069		
	3	0.002915	0.832745	0.578864	0.761536	0.674500	
	4	0.004080	0.949805	0.691702	0.881063	0.877943	0.750770

Table 7. The probabilities for false rejection $P_N(0,1;0,1,n,a)$, and for critical systematic error detection $P_N(m_c,1;0,1,n,a)$, $P_M(m_c,1;0,1,n,l_M)$, $P_S(m_c,1;0,1,n,l_S)$ and $P_R(m_c,1;0,1,n,l_R)$ for $2 \leq n \leq 4$, $1 \leq k \leq n$, $a \in \{1,2,3,4,6,8,12,16\}$, and $m_c = 2.63$, given that $P_N(0,1;0,1,n,6) = P_M(0,1;0,1,n,l_M) = P_S(0,1;0,1,n,l_S) = P_R(0,1;0,1,n,l_R)$.

a	n	$P_N(0,1;0,1,n,a)$	$P_N(2.63,1;0,1,n,a)$	$P_M(2.63,1;0,1,n,l_M)$	$P_S(2.63,1;0,1,n,l_S)$	$P_R(2.63,1;0,1,n,l_R)$
1	2	0.114545	0.965929	0.983601	0.114545	0.114545
	3	0.120814	0.993068	0.998631	0.120814	0.120814
	4	0.101483	0.998901	0.999849	0.101483	0.101483
2	2	0.046078	0.924509	0.957097	0.046078	0.046078
	3	0.041528	0.973308	0.993943	0.041528	0.041528
	4	0.040159	0.993895	0.999309	0.040159	0.040159
3	2	0.031890	0.893212	0.941424	0.031890	0.031890
	3	0.024233	0.959429	0.989093	0.024233	0.024233
	4	0.029011	0.993239	0.998919	0.029011	0.029011
4	2	0.022235	0.854991	0.923109	0.022235	0.022235
	3	0.014565	0.942030	0.982311	0.014565	0.014565
	4	0.014786	0.985589	0.997545	0.014786	0.014786
6	2	0.016204	0.835627	0.904620	0.016204	0.016204
	3	0.009234	0.911144	0.974037	0.009234	0.009234
	4	0.010381	0.975338	0.996401	0.010381	0.010381
8	2	0.007229	0.767347	0.847674	0.007229	0.007229
	3	0.005146	0.881898	0.959879	0.005146	0.005146
	4	0.005599	0.967433	0.993432	0.005599	0.005599
12	2	0.003311	0.664467	0.780938	0.003311	0.003311
	3	0.005211	0.872698	0.960228	0.005211	0.005211
	4	0.003848	0.956170	0.990864	0.003848	0.003848
16	2	0.002574	0.634268	0.757440	0.002574	0.002574
	3	0.002915	0.832745	0.941846	0.002915	0.002915
	4	0.004080	0.949805	0.991309	0.004080	0.004080

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